FINITE-AMPLITUDE CYLINDRICAL AND SPHERICAL WAVES IN WEAKLY DISPERSIVE MEDIA

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Problems related to excitation and propagation of nonlinear cylindrical and spherical waves in media with weak dispersion occur in many branches of theoretical and applied physics. Excitations of this kind are observed, for example, in plasmas [1-3], on the surface of a shallow fluid (tsunami waves, etc.) [4, 5], in gas-dynamics [6], in nonlinear lattices, etc. The evolution of the corresponding one-dimensional processes, described by planar waves by means of the Korteweg–de Vries (KdV) equation, has been studied in much detail [7]. In recent years several attempts have been made at generalizing this equation for describing non-planar waves with axial or central symmetry [5, 8-11]. Several partial solutions were obtained of the generalized KdV equation, describing quasistationary solitary pulses and solitons [2, 3, 5, 8-11], and were observed in a number of experiments [1-4]. The present paper is devoted to further study of cylindrical and spherical waves; several new approximate solutions are obtained, taking into account loss effects in the medium, several cases involving some prevailing factors (nonlinearity, dispersion, geometric divergence) are worked out, and the results obtained are compared with experimental data.

1. The boundary-value problem for the KdV equation, generalized to the cases of motion with axial and central symmetry, can be represented in dimensionless variables in the form [8, 9]

$$\begin{gathered}\partial u/\partial r + \mu u\partial u/\partial\tau + \varepsilon^2\partial^3 u/\partial\tau^3 + \chi u + S\varkappa u/(1+\varkappa r) = 0,\\ u(0,\tau) = f(\tau),\end{gathered} \tag{1.1}$$

where $\tau = r - t$, t is time, r is the radial coordinate measured from the boundary of the surface at distance $\varkappa^{-1}$ from the center, u is the mass velocity of the medium, μ, ε, χ are small parameters characterizing the nonlinearity, dispersion, and low-frequency dissipation, respectively, S is a coefficient having the values 0, $1/2$, 1 for the cases of planar, axial, and central symmetry, respectively, and f is a given positive finite function of amplitude unity. In this equation we restrict ourselves to models of frequency-independent losses.

We derive the integral consequences of Eq. (1.1). For this we multiply Eq. (1.1) by u^{k-1} ($k=1, 2$) and integrate the result over from $-\infty$ to $+\infty$; taking then into account the condition of rest at infinity, we obtain

$$I_k(r) = I_k(r_*)\varphi^k(r, r_*),\quad k = 1, 2. \tag{1.2}$$

The quantity $I_k = \int\limits_{-\infty}^{\infty} u^k(\tau, r)\,d\tau$ is the total momentum of motion of the medium for $k=1$, and the total energy for $k=2$. In what follows this quantity plays an important role in constructing approximate solutions and explaining the nature of their r dependence. The function $\varphi(r, r_*)$ is defined by the equation $\varphi(r, r_*) = [1+\varkappa(r-r_*)]^{-S}\cdot e^{-\chi(r-r_*)}$, where r_* is some fixed value of separation. We note that for $\chi=0$ Eq. (1.1) in the cylindrical case $S=1/2$ is similar to the ordinary KdV equation, it possesses an infinite set of integrals of motion [12], and can be represented in the Lax form with corresponding LA pairs [13].

We replace variables in the KdV equation (1.1). We introduce new variables U, x by means of the relations $U = u\varphi^{-1}(r, 0)$

$$x = \mu\int\limits_0^r \varphi(r', 0)\,dr' = \mu\begin{cases} r, & S=0,\\ \frac{2}{\varkappa}\left(\sqrt{1+\varkappa r}-1\right), & S=\frac{1}{2},\\ \frac{1}{\varkappa}\ln(1+\varkappa r), & S=1\end{cases}$$

for $\chi=0$, and

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$$x=\mu\begin{cases}\frac{1}{\chi}\left(1-e^{-\chi^{2}}\right), & S=0,\\ \sqrt{\frac{\pi}{\varkappa\chi}}\,e^{\frac{\chi}{\varkappa}}\left\{\Phi\sqrt{\frac{\chi}{\varkappa}(1+\varkappa r)}\right]-\Phi\left[\sqrt{\frac{\chi}{\varkappa}}\right]\Big\}, & S=\frac{1}{2},\\ \frac{1}{\varkappa}\,e^{\frac{\chi}{\varkappa}}\left\{\mathrm{Ei}\left[-\frac{\chi}{\varkappa}(1+\varkappa r)\right]-\mathrm{Ei}\left[-\frac{\chi}{\varkappa}\right]\right\}, & S=1\end{cases}$$

for $\chi \neq 0$. Here Φ is the probability integral, and Ei is the integral exponential function. In the new variables Eq. (1.1) is reduced to the form

$$\frac{\partial U}{\partial x}+U\frac{\partial U}{\partial \tau}+\frac{1}{\beta^{2}(x)}\frac{\partial^{3}U}{\partial \tau^{3}}=0,\quad \beta=\frac{\sqrt{\mu\varphi(x,0)}}{\varepsilon}. \tag{1.3}$$

It hence follows that the effective similarity criterion, expressed by the ratio of nonlinear to dispersion effects, is in this case the quantity $\beta^2(x)$, which is not constant. This fact causes a strong dependence of the solution of the boundary-value problem on distance, unlike the problem of planar symmetry, where the structure of the wave pattern is determined by the constant β^2 [7].

2. Consider first the case in which $\beta^2 \ll 1$ at the boundary of the surface, and let for simplicity $\chi = 0$. The nonlinear term in Eq. (1.3) can then be neglected, and the general solution of the linearized equation (1.3) is [7]

$$u(r,\tau)=\pi^{-\frac{1}{2}}(3\varepsilon^{2}r)^{-\frac{1}{3}}(1+\varkappa r)^{-S}\int\limits_{-\infty}^{\infty}\mathrm{Ai}\left[\frac{\tau-\tau'}{(3\varepsilon^{2}r)^{1/3}}\right]f(\tau')\,d\tau'. \tag{2.1}$$

As shown in [7], for asymptotically large r, τ the solution (2.1) for initial perturbations with nonvanishing area is expressed in terms of the Airy function:

$$u(r,\tau)\sim r^{-\left(S+\frac{1}{3}\right)}\,\mathrm{Ai}\left(\frac{\tau}{r^{1/3}}\right), \tag{2.2}$$

whence it follows that the wave amplitude decreases as $\sim r^{-5/6}$ in the cylindrical case and $\sim r^{-4/3}$ in the spherical case, and the characteristic length increases as $\sim r^{1/3}$, while the first incoming wave has the largest amplitude. We note that the approximate solution (2.2) of the linearized equation (1.3) can be obtained from simple considerations of dimensionality. For this it is necessary to use the conservation law (1.2) with k = 1. Using the energy conservation law with k = 2, we obtain a different self-similar solution whose duration varies with distance in the same away as for Eq. (2.2), and whose amplitude falls off according to the weaker law $\sim r^{(S+1/6)}$. On the basis of these data one can obtain the law of variation of the parameter β^2 with distance, determined by the local value of the product of the wave amplitude by the square of its intensity at each moment of time. Thus, for the first-type solution $\beta \sim r^{-(1/2)(S-1/3)}$, and for the second $\beta \sim r^{-(1/2)(S-1/2)}$. As seen from the expression for β, in both cases for $S \neq 0$ this parameter does not increase with distance, so that an initially linear wave always remains linear. We note that in the planar case S = 0 the parameter β increases with distance for both types of solution, so that a wave with arbitrarily small amplitude and nonvanishing area always becomes nonlinear. For a converging wave this parameter increases with decreasing r, except for the case $S = 1/2$ when the second type of solution leads to an increasing role of nonlinearity which, starting at some distance, becomes substantial for further description of the wave. Analytic estimates [14] show that usually the amplitude of cylindrical waves decreases near the perturbation source as $\sim r^{-2/3}$, in agreement with the energy conservation law, while at large distances the solution (2.2), decreasing as $\sim r^{-5/6}$, dominates.

It must be noted that the results obtained here refer to the linearized KdV equation, which is, however, not always valid for describing axially symmetric waves. The invalidity of this equation at short distances from the center is obvious by the strong enhancement of the role of the last term in Eq. (1.1), while by the derivation of (1.1) this term is relatively small. This equation also becomes invalid at large distances, since it describes the leading part of the wave (the low-frequency spectral region), containing a relatively small portion of the energy of the whole wave. Most of the energy is contained in the high-frequency wave train, and is usually described by an asymptotic calculation of Fourier–Bessel integrals, without using the single-wave approximation and being an exact solution of the original problem in the linear statement.

Thus, the results obtained here can be considered as intermediate asymptotic, valid at moderate distances from the center. As an example, Fig. 1 shows the result of a numerical calculation of circular waves from an axially symmetric source, having the form $u(0, r) = U_0 e^{-(r/a)^2}$, $a = 2$. The calculation was performed on the basis of the linearized equations of shallow water by reduction to Fourier–Bessel integrals, but at some distance from the center, characteristic of a perturbation scale much smaller than the radius of the circular

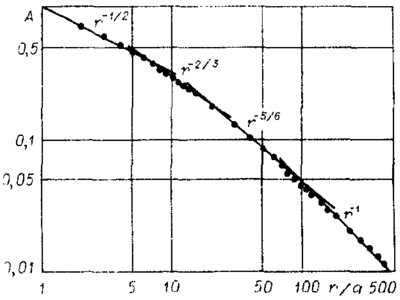

Fig. 1

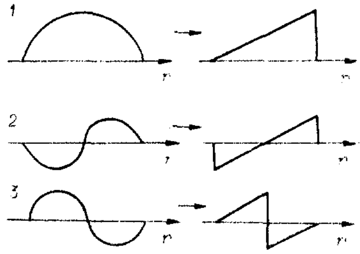

Fig. 2

wave, its evolution can be described by Eq. (1.1) with $\chi = 0$. As seen from Fig. 1, initially (up to $r \simeq 2.5a$), when dispersion can be neglected, the wave amplitude decreases as $\sim r^{-1/2}$. Then, from $r \simeq 2.5a$ to $r \simeq 8a$ the amplitude varies as $r^{-2/3}$, in agreement with [14]. At $r \simeq 8a$ a transition occurs to the law $r^{-5/6}$, which is valid up to $r \simeq 60a$. At long distances, finally, the wave head decreases as $\sim r^{-1}$. Thus, it is seen that the decrease in the wave amplitude at the portion from $r \simeq 1.5a$ to $r \simeq 60a$ can be described within the single-wave equation (1.1) and is explained by the laws derived in this section.

3. Consider now the case of a boundary-value problem (1.1), for which nonlinear effects dominate over dispersion effects at short distances from the surface boundary. The dispersion term in (1.1) can then be neglected, and within the remaining equations, describing simple waves with damping and divergence, one can find exact solutions of the form

$$u(\tau, r) = f\left[\tau - \frac{\mu u}{\varphi(r, 0)} \int_0^r \varphi(r', 0)\, dr'\right], \tag{3.1}$$

where the function f is given by the boundary condition. The distance dependence of the wave amplitude is found from the relation

$$u e^{\varkappa r}(1 + \varkappa r)^S = \mathrm{const}, \tag{3.2}$$

which follows from (1.2) for k = 1. As seen from Eq. (3.2), the amplitude of weak waves varies with distance in the same way as linear waves [strongly nonlinear simple waves are not described by Eq. (1.1), and therefore their amplitude varies according to a different law [6]]. The dependence of the wave amplitude (3.1) on u leads to a change in its form. The calculation of the distance at which a bore wave is formed and finding the height attenuation laws are performed in the same way as in the problem of sound shock waves in gases and liquids [6], therefore we provide here the main equations without derivation. Three types of bore waves are possible, as illustrated in Fig. 2, depending on the shape of the initial perturbation. The following asymptotic equations are valid for the amplitudes of the first two types:

$$u \sim r^{-3/4}, \ \Lambda \sim r^{1/4},$$

while for the third type the variation laws of u and Λ are different:

$$u \sim r^{-1}, \ \Lambda = \mathrm{const},$$

where Λ is the characteristic size of the bore wave. The experimental data obtained in [15] for charge detonation of 1000 kg of litho-trotyl in shallow water (with depth of order 50 cm) agree quite accurately with Eq. (3.2) for cylindrical waves ($S = 1/2$) at the surface of the liquid without accounting for χ. Figure 3 shows the shapes of surface elevations, selected from [15], at distances 8, 11.6, 15.3, 22.9, and 30.8 m from the detonation epicenter; it is seen that the wave shape is near a triangular bore wave, and by the data provided in [15] its amplitude changes as $\sim r^{-0.75}$.

Due to the decreasing perturbation amplitude of diverging waves nonlinear effects may become of the same order as dispersion effects at some distance (if the wave scale varies sufficiently slowly in comparison with its amplitude). In this case wave propagation is described by the complete equation (1.1), and the structure of the wave pattern acquires a quasisoliton character if the last two terms in (1.1) are sufficiently small.*

*Qualitatively the change of regime follows already from (1.3), since the parameter $\beta^2(x)$ is not constant. To determine the distance at which there is a transition from nonlinear to dispersion nature of the solution it is necessary to have more accurate knowledge of the parameter $\beta^2(x)$, based on the solution already found.

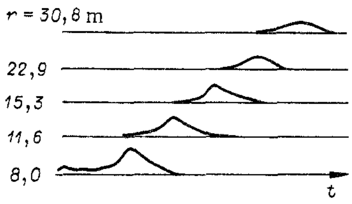

Fig. 3

4. Taking into account the nonlinearity and dispersion, separate classes of exact solutions of Eq. (1.1) for $\chi = 0$ and $S = 1/2$ were found in [12, 13]. The nature of these solutions is quite similar to the N-soliton solutions corresponding to the KdV equation in the planar case [7], but their expressions are too awkward and inconvenient for practical calculations. Another approach for finding approximate solutions of (1.1) was used in [2, 3, 5, 8-10, 16] and was based on the similarity of the solution of (1.1) to a planar soliton if the two last terms in (1.1) are sufficiently small. Extensive experimental studies and numerical calculations [1-5], carried out for both cylindrical and spherical waves, showed that in both cases a decay of an initial perturbation of arbitrary form into a number of solitons is observed, similar to what happens in the planar case [7]. The soliton amplitude decreases with distance due to divergence and dissipation.

Many authors attempted to obtain theoretically the variation law of soliton amplitude with distance without account of dissipation. Contradictory data were obtained in this case: According to [5, 8, 9] the soliton amplitude changes as $\sim r^{-S}$, while, according to [10, 16], the amplitude variation law is stronger $\sim r^{-(4/3)S}$. In our opinion this contradiction is explained by the fact that in the first group of studies the variation of the soliton duration with distance, related to its amplitude, was not taken into account. To find the variation laws of the soliton parameters with distance we assume, as was already mentioned, that the last two terms in (1.1) are small, and the solution is of the same form as in the planar case, but with slowly varying parameters, the amplitude A and the duration Λ:

$$u = A(r)\,\mathrm{sech}^2 \frac{\tau - \int\limits_0^r \frac{\mu A}{3}\, dr}{\Lambda(r)}, \tag{4.1}$$

where $\Lambda(r) = \sqrt{12\varepsilon^2/\mu A(r)}$. Using the energy conservation law (1.2) and the relation between Λ and A, we obtain

$$A \sim r^{-(4/3)S} e^{-(4/3)\chi r}, \quad \Lambda \sim r^{(2/3)S} e^{(2/3)\chi r}. \tag{4.2}$$

It is interesting to note that the same law of decreasing field with distance follows from the exact solutions [12, 13], as well as from the self-similar solutions found in [11]. More rigorous solutions of type (4.1), (4.2), found by means of asymptotic expansions, were obtained in [16]. The laws of variation of the soliton parameters (4.2) are in good agreement with the numerical data represented in Fig. 4, where we show the variation of soliton amplitude with distance, calculated on the basis of Eq. (1.1) for $\chi = 0$. (The straight line $\sim r^{-2/3}$ corresponds to cylindrical, and the straight line $\sim r^{-4/3}$ to spherical solitons. For comparison we show by the dashed line the variation law of amplitudes of linear waves without dispersion, and the points are 1–[5], 2–[8], 3–[9]). There is also good agreement between Eqs. (4.1), (4.2) and the experimental data obtained with plasma solitons. Figure 5 shows the distance dependence of amplitudes of spherically diverging solitons (the points 1 are from [2], and 2 from [3]). The electromodeling performed by us of cylindrically diverging solitons by means of nonlinear two-dimensional LC-lattices led to coincidence of the data obtained on variations of soliton amplitudes

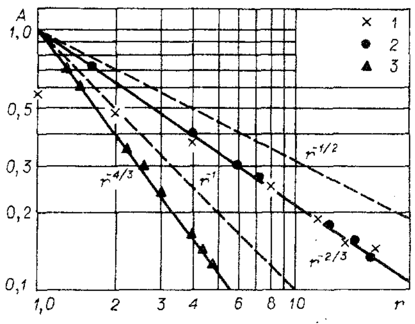

Fig. 4

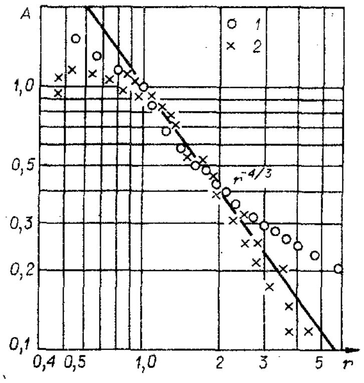

Fig. 5

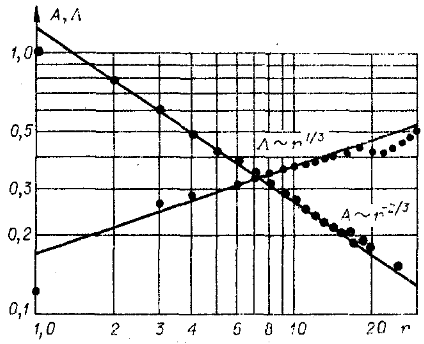

Fig. 6

and durations with the calculated equations. Figure 6 shows the experimental data for dimensionless soliton amplitudes (normalized to the amplitude of the input pulse) and durations (in microseconds). Unfortunately, the other experimental studies observing solitons in plasma [1] and in water [4] have a more qualitative nature. We also note that in many experimental studies the term "soliton" is applied to arbitrary solitary waves not described by Eq. (1.1), which often leads to confusion, since the parameters of these solitary waves vary with distance according to laws different than (4.1), (4.2). The generalized damping soliton (4.1) is not formed independently, and in the propagation process it emits a wave packet ("tail") which can be found in the following approximation in constructing an asymptotic solution [16]. In the planar case $S=0$, $\chi \neq 0$ the "tail" structure was studied in detail in many studies in recent years. Here we restrict ourselves to the expression for a pulse "tail," which can be found from (1.2) for $k=1$:

$$I_* = 2\sqrt{\frac{12\varepsilon^2 A_0}{\mu}}\, e^{-(2/3)\chi r}(1+\varkappa r)^{-(2/3)S}\left[e^{-\frac{\varkappa r}{3}}(1+\varkappa r)^{-S/3}-1\right],$$

where A_0 is the soliton amplitude at $r=0$. For large r the pulse "tail" equals the soliton pulse in absolute value and opposes it in sign, decreasing with distance according to the law $I_* \sim -e^{-(2/3)\chi r}\, r^{-(2/3)S}$.

The quasisoliton solutions described are valid at restricted distances due to the impossibility of simultaneously "balancing" three factors: nonlinearity, dispersion, and divergence. Finally, it is important to note that in the absence of dissipation ($\chi=0$) the solution of the zeroth approximation (4.1), (4.2) in the cylindrical case remains valid until the correction due to the following approximation becomes sufficiently large. In this case the ratio of the last term in (1.1) to the dispersion (or nonlinear) term remains small if it is initially small. In the spherical case the solution (4.1), (4.2) can become unsuitable much earlier, since the ratio mentioned above increases with distance. The boundary of the applicability region of this solution is estimated as $r_0 \sim (1/\varepsilon)(12/\mu)^{3/2}$. For $r \gg r_0$ the solution becomes linear, and is described by the equations of Sec. 2.

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MIXING OF A CONTACT BOUNDARY RETARDED BY STATIONARY SHOCK WAVES

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The phenomenon of turbulent mixing of the interface between two gases of different densities retarded by plane stationary shock waves moving from the light gas into the heavy one was discovered experimentally in [1].

It is shown below that within the framework of the semiempirical models of [1–3] this phenomenon is determined by the size of the initial perturbations – the roughness of the interface. If the characteristic size of these perturbations approaches zero, then the width of the mixing region also approaches zero. This phenomenon is explained by the δ-function character of the acceleration.

If the acceleration varies smoothly, such as constantly, then mixing will always develop, even with infinitely small roughness. The analytical dependence of the width of the mixing region on the initial roughness is presented.

The interface of the gases (liquids) is unstable against small perturbations if the acceleration is directed from the light to the heavy gas. This instability develops for sufficiently small coefficients of viscosity and surface tension.

In the semiempirical models of [1–3] it is assumed that turbulent mixing develops simultaneously with the action of the acceleration, although actually the presence of viscosity and surface tension leads to the appearance of a finite time interval during which a gradual transition to turbulent motion occurs.

The known self-similar solutions [3–5] were obtained under the assumption of smallness of the initial perturbations. Actually, these perturbations may not be small. The law according to which the emergence into a self-similar solution with constant acceleration occurs is established below. A mild "forgetting" of the initial irregularities of the surface was unexpectedly discovered.

1. Approximate Model

We will consider a diffusional model of turbulent mixing in the approximate formulation of [5]: The fluids are incompressible, while the turbulent velocity v is assumed to be a function of time only. Then the process of turbulent mixing will be described by two equations for two unknowns (the density ρ of the mixture and the characteristic turbulent velocity v),

$$\frac{\partial \rho}{\partial t} = lv\frac{\partial^2 \rho}{\partial x^2}; \tag{1.1}$$

$$\frac{1}{2}\frac{dv^2}{dt} + \frac{\nu}{\alpha}\frac{v^3}{L} = \alpha v\omega^2, \tag{1.2}$$

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